INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N ° 3762

Septembre 1999

THÈME 2

Rapport de recherche

ISSN 0249-6399

Generalized Hamming Weight as a Weight Function

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Thème 2 — Génie logiciel
et calcul symbolique
Projet CODES

Rapport de recherche n° 3762 — Septembre 1999 — 24 pages

Abstract: We prove that the weight function of a linear code, that is, an integer function defined on the vector space of messages, uniquely determines the code up to equivalence. We propose a natural way to extend the r-th generalized Hamming weight, that is, a function on r-subspaces of a code, to a function on the r-th exterior power of the code. Using this, we show that for any linear code C and any integer r not greater than the dimension of C, another code C' exists whose weight distribution corresponds to a part of the generalized weight spectrum of C from the r-th weights to the k-th. In particular, the minimum distance of C' is proportional to the r-th generalized weight of C.

Key-words: code, equivalence, generalized Hamming weight, weight enumerator, generalized spectrum

(Résumé : tsvp)

Supported in part by the Russian Fundamental Research Foundation (project No. 96-01-01378)

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Poids de Hamming généralisé vu comme une fonction de poids

Résumé : La fonction de poids d'un code linéaire est une fonction à valeurs entières, définie sur l'espace des messages, qui à un message fait correspondre le poids du message encodé. Nous prouvons que cette fonction détermine de façon unique le code auquel elle est rattachée, à une équivalence près.

Le $r$ième poids de Hamming généralisé d'un code linéaire donné est une fonction à valeurs entières sur les sous-espaces de dimension r du code. Nous proposons une extension naturelle de cette fonction, en une fonction sur le $r$ième produit extérieur du code. Nous montrons ensuite que pour tout code linéaire C et tout entier r, inférieur ou égal à la dimension de C, un autre code existe, soit C', dont la distribution des poids est, pour une part, le spectre de Hamming généralisé de C. En particulier la distance minimale de C' est proportionnelle au $r$ième poids généralisé de C.

Mots-clé : code, équivalence, poid de Hamming généralisé, énumérateur des poids, spectre généralisé

1 Introduction

Let us begin with a simple question. Let a linear $[n,k]_q$ code C be given, i.e., q^k messages are transmitted with the help of length-n words. Since the code is linear, it is natural to consider that the set of messages also has a linear structure, i.e., is identified with the space $\mathbb{F}_q^k$. Assume now that, for each message, we only know the *weight* of the codeword by which this message is transmitted but not the codeword itself. If we know these weights only, is it possible to reconstruct the code uniquely up to equivalence (i.e., permutation of coordinates and multiplying each coordinate by a constant)? A simple observations shows that the answer is positive (Sec. 2).

Thus, given all the weights, i.e., a nonnegative function $\mathbb{F}_q^k \to \mathbb{Z}$ (more precisely, a function $\mathrm{wt}\colon \mathbb{P}^{k-1} \to \mathbb{Z}$; here, $\mathbb{P}^{k-1} = \mathbb{P}\mathbb{F}_q^k \cong \mathbb{P}C$), one can reconstruct a code according to a certain rule (see (4) below) up to equivalence or, which is the same, reconstruct the projective system that corresponds to the code. In its turn, a projective system (multiset) is a nonnegative function $\nu\colon \mathbb{P}C^* \to \mathbb{Z}$, where $\nu(h)$ is the multiplicity of a point $h \in \mathbb{P}C^*$. In other words, there exists a transform which sends a function wt to a function ν.

If we apply this transform to an arbitrary function $\widetilde{\mathrm{wt}}\colon \mathbb{P}^{k-1} \to \mathbb{Z}$ which is not a weight function of any code, then the function $\widetilde{\nu}\colon (\mathbb{P}^{k-1})^* \to \mathbb{Q}$ obtained according to (4) will not necessarily be integer and nonnegative. However, with an appropriate choice of constants a and b, the function $a\widetilde{\nu} + b$ becomes integer and nonnegative, i.e., defines a projective system (code) whose weights are easily computed from $\widetilde{\mathrm{wt}}$, a, and b. This reasoning can be viewed as a way of constructing linear codes. Of course, for an arbitrary initial function $\widetilde{\mathrm{wt}}$, the multiplicities thus obtained can be very large and the code can be by far not good, but with an apt choice of $\widetilde{\mathrm{wt}}$ the construction may be of interest.

Some generalizations of this idea are discussed in Sec. 3.

In Sec. 4, we give an example of such a function $\widetilde{\mathrm{wt}}$ the study of which stimulated this research. Namely, as a weight function $\widetilde{\mathrm{wt}}$ we consider the generalized Hamming weights (support weights) of an arbitrary fixed linear code C. It should immediately be noted that, according to the aforesaid, the function $\widetilde{\mathrm{wt}}$ must be defined on some projective space, whereas the rth generalized weight is a function on r-subspaces of a code C, i.e., on the Grassmann variety $G(r,C)$. However, there exists a natural embedding of $G(r,C)$ in a projective space—the Plücker embedding $G(r,C) \hookrightarrow \mathbb{P}^{\binom{k}{r}-1} = \mathbb{P}\Lambda^r C$ (here, $k = \dim C$ and $\Lambda^r C$ is the rth exterior power of C)—and the question is how to extend the generalized-weight function from $G(r,C)$ to the entire space $\mathbb{P}\Lambda^r C$ in an appropriate way. In Sec. 4 we suggest such an extension which, on one hand, seems to be quite natural and deserves being investigated and, on the other hand, makes it possible to explicitly compute $\widetilde{\nu}$ according to (4). Thus, given a code C, we obtain a new code $\boldsymbol{C}$ (with multiplicities $\widetilde{\nu}$ and weights $\widetilde{\mathrm{wt}}$) whose minimum distance is expressed via the rth generalized weight of C and whose weight spectrum is expressed via the generalized weight spectrum of C (precisely, via the part of the generalized spectrum from the rth weights to the kth).

Some of the results of this paper are in brief presented in [12].

A part of this work was done when the author was visiting the Institut de Mathématiques de Luminy, Marseille (February 1999) and INRIA Rocquencourt, Paris, project CODES (August/September 1999). He would like to use this opportunity to thank the IML and INRIA for the hospitality.

2 Weights and multiplicities

Let C be a fixed linear $[n,k]_q$ code. We assume that C is nondegenerate, i.e., has no identically zero coordinates; in other words, its effective length equals its length. Consider the (Hamming) weight function on C. Since the weight of the all-zero word always equals zero and proportional words have equal weights, this function is in fact defined on the projective space $\mathbb{P}^{k-1} = \mathbb{P}C$. Thus, the object of our study is the function $\mathrm{wt}\colon \mathbb{P}C \to \mathbb{Z}$. In the previous section, we put the problem of the reconstruction of C up to equivalence given this function wt. The coordinates (columns of the generator matrix) of a code are elements of the space C^* of linear functions on C. In the space $\mathbb{P}C$, to each coordinate (since C is nondegenerate) corresponds a hyperplane—the zero set of this coordinate. To proportional coordinates corresponds the same hyperplane. Therefore, it is natural to assign to each hyperplane its multiplicity ν and consider the function $\nu(h)$ on the space of hyperplanes, putting $\nu(h) = 0$ if h does not correspond to any coordinate.

The weight of a codeword c is the number of coordinates that do not vanish on c, i.e., the number of coordinate hyperplanes such that c lies *outside* them, counted with multiplicities. Therefore, by the definition of the weight,

$$\mathrm{wt}(c) \stackrel{\mathrm{def}}{=} \sum_{h \not\ni c} \nu(h). \tag{1}$$

If the hyperplanes h are viewed as points of the dual projective space $\mathbb{P}C^*$, we obtain a projective multiset defined by the mapping ν, or (which is the same) a projective $[n,k]_q$ system. As is well known [1, Sec. 1.1.2] (for details, see [2]), there exists a natural one-to-one correspondence between equivalence classes of nondegenerate $[n,k]$ systems and equivalence classes of nondegenerate linear $[n,k]$ codes. Therefore, the problem of the reconstruction of a code C up to equivalence given a function wt is the problem of the reconstruction of a function $\nu\colon \mathbb{P}C^* \to \mathbb{Z}$ given a function $\mathrm{wt}\colon \mathbb{P}C \to \mathbb{Z}$. In other words, if we regard the formula (1) as a transform which sends a function ν to a function wt, we put the question on the inverse transform. The answer to this question is rather simple and is due to the following observation.

Let us consider $\sum_{c \in \mathbb{P}C} \mathrm{wt}(c)$. According to (1),

$$\sum_{c \in \mathbb{P}C} \mathrm{wt}(c) = \sum_{c \in \mathbb{P}C} \sum_{h \not\ni c} \nu(h) = \sum_{h} \nu(h) \sum_{c \notin h} 1 = q^{k-1} \cdot \sum_{h} \nu(h) \tag{2}$$

(here, q^{k-1} is the number of points of $\mathbb{P}C$ lying outside a hyperplane, i.e., the number of points of an affine space of dimension $k-1$). Considering $\sum_{c \in h_0} \mathrm{wt}(c)$, where h_0 is a fixed hyperplane, we find

$$\sum_{c \in h_0} \mathrm{wt}(c) = \sum_{c \in h_0} \sum_{h \not\ni c} \nu(h) = \sum_{h \neq h_0} \nu(h) \sum_{c \notin h \setminus h_0} 1 = q^{k-2} \cdot \sum_{h \neq h_0} \nu(h) \tag{3}$$

(here, q^{k-2} is the number of points of $h_0 \cong \mathbb{P}^{k-2}$ lying outside a hyperplane in h_0, i.e., the number of points of an affine space of dimension $k-1$). Now, multiplying (3) by q and subtracting this from (2), we immediately obtain (substituting h for h_0) the following formula.

Proposition 1 [12] (inversion formula).

$$\nu(h) = \frac{\sum\limits_{c} \mathrm{wt}(c) - q \sum\limits_{c \in h} \mathrm{wt}(c)}{q^{k-1}}. \tag{4}$$

Note that we may apply formula (4) to an arbitrary function $\widetilde{\mathrm{wt}}\colon \mathbb{P}^{k-1} \to \mathbb{Z}$ which need not be an actual weight function of any linear code. Then the "multiplicities" $\widetilde{\nu}$ obtained according to the rule (4) will not necessarily be integer but may take rational values including negative ones.

Problem 1. What is the *minimum* (in any sense) set of conditions on $\widetilde{\mathrm{wt}}$ under which all $\widetilde{\nu}(h)$ obtained according to (4) are nonnegative integers?

Of course, a function $\widetilde{\nu}\colon (\mathbb{P}^{k-1})^* \to \mathbb{Q}$ can be made integer and nonnegative by adding $-\min\limits_{h} \widetilde{\nu}(h)$ and multiplying the result by the common denominator of the numbers obtained. This operation corresponds to a linear transformation of the function $\widetilde{\mathrm{wt}}$ since the transforms (1) and (4) are linear in ν and wt respectively. In other words, one can always find constants a and b such that $a\widetilde{\mathrm{wt}}+b$ is an actual weight function, i.e., corresponds to a linear code (possibly, with repetitions). This idea may be considered as a way of constructing linear codes. Surely, for an *arbitrarily* chosen $\widetilde{\mathrm{wt}}$, the code thus obtained will most probably be bad since the number of repetitions can be large. However, for a suitable choice of a "good" function $\widetilde{\mathrm{wt}}$, one may expect good results. In the present paper, we are mainly interested in other aspects.

Remark 1. If we construct a code in such a way, we must bear in mind the possibility of a dimension drop, that is, a case where the projective system obtained is degenerate, i.e., the support of the function $\widetilde{\nu}$ lies in a hyperplane. However, this can easily be determined via the function $\widetilde{\mathrm{wt}}$ since in this case a point (corresponding to this hyperplane) with zero "weight" should exist.

Moreover, due to the linearity of (1) and (4) in ν and wt respectively, "degenerate" functions ν are also worth studying since a nondegenerate $\widetilde{\nu}$ can be represented as a sum of degenerate ones. Thus, in the following simple example which we need below, the function ν is degenerate.

Example 1. Let h^α be a plane of codimension α in $\mathbb{P}^{k-1}$. As a weight function, consider the indicator function of the complement to h^α, i.e.,

$$\mathrm{wt}(c) = \begin{cases} 1, & c \notin h^\alpha, \\ 0, & c \in h^\alpha. \end{cases}$$

What is the function ν corresponding to this wt? According to (4), let us compute

$$\sum_c \mathrm{wt}(c) = \theta_{k-1} - \theta_{k-\alpha-1} = q^{k-\alpha}\theta_{\alpha-1},$$

where $\theta_a = \dfrac{q^{a+1}-1}{q-1}$ is the number of points of $\mathbb{P}^a$ and we have used the formula $\theta_a - \theta_b = q^{b+1}\theta_{\alpha-1}$. Furthermore, according to (4) we have to compute $\sum_{c\in h} \mathrm{wt}(c)$. Here two cases are possible: $h \supset h^\alpha$ and $h \cap h^\alpha = h^{\alpha+1}$. For these cases, we find

$$\begin{aligned} \sum_{c\in h\supset h^\alpha} \mathrm{wt}(c) &= \theta_{k-2} - \theta_{k-\alpha-1} = q^{k-\alpha}\theta_{\alpha-2}, \\ \sum_{c\in h\not\supset h^\alpha} \mathrm{wt}(c) &= \theta_{k-2} - \theta_{k-\alpha-2} = q^{k-\alpha-1}\theta_{\alpha-1}. \end{aligned}$$

Hence,

$$\nu(h) = \begin{cases} q^{1-\alpha}, & h \supset h^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the condition that a hyperplane $h \subset \mathbb{P}C$ contains a fixed plane h^α is equivalent to the condition that the corresponding point $h \in \mathbb{P}C^*$ lies in a plane $h_{\alpha-1}$ of (projective) dimension $\alpha - 1$. Thus, ν as a function on $\mathbb{P}C^*$ is defined by the conditions

$$\nu(h) = \begin{cases} q^{1-\alpha}, & h \in h_{\alpha-1}, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., up to a factor, is the indicator function of $h_{\alpha-1}$. Surely, from the point of view of projective systems, this answer is trivial—if we consider a projective system that consists of the points of $h_{\alpha-1}$, then the corresponding weights are equal to $q^{\alpha-1}$ and lie everywhere outside h^α.

3 Several remarks

In this section, we discuss possible generalizations of the construction proposed.

3.1. Consider the transform $R\colon \nu \mapsto \mathrm{wt}$ defined by (1).

As Boguslavsky has noted [1, 2], the function

$$n - \mathrm{wt}(c) = \sum_{h \in c} \nu(h) =: \widehat{\nu}(c) \tag{5}$$

is the discrete analog of the *Radon transform* [13], which in the continuous case is defined as follows: for a rapidly decreasing function $f\colon \mathbb{R}^k \to \mathbb{R}$, its Radon transform $\widehat{f}(H)$ is the function defined on the set of hyperplanes in $\mathbb{R}^k$ given by the formula

$$\widehat{f}(H) = \int_H f\, d\mu_H,$$

where μ_H is the natural measure on a hyperplane. Generalization to the case of two homogeneous spaces with an incidence relation can be found in [7, 8].

Since we consider ν as a function on $\mathbb{P}C^*$, the sum $\sum_{h \in c}$ in (5) is precisely the integral over a hyperplane in $\mathbb{P}C^*$. In this sense, the transform $R\colon \nu \mapsto \mathrm{wt}$ defined by (1) is an "anti-Radon" transform since the integral is taken over the complement to a hyperplane, and the formula (4) defines the inverse transform $R^{-1}\colon \mathrm{wt} \mapsto \nu$. Note that in the continuous case the question on the inverse Radon transform is not trivial since the inverse transform is an integral transform which involves a certain differential operator applied to $\widehat{f}$.

3.2. Let $D_r \subset C$ be an r-subcode of C. Recall [16] that the *r-th generalized weight* (also known as the *support length* or *Hamming norm*) wt_r of a subcode D_r is the number of coordinates that do not vanish on D_r. In the space $\mathbb{P}C^*$ to an r-subcode corresponds a plane h^r of codimension r; thus, similarly to (1), we can write [15]

$$\mathrm{wt}_r(h^r) \stackrel{\mathrm{def}}{=} \sum_{h \notin h^r} \nu(h). \tag{6}$$

(This formula can be interpreted as the r-plane discrete anti-Radon transform.) For $r = k$ and $r = k-1$, it obviously follows from the definition that $\mathrm{wt}_k \equiv n$ and $\mathrm{wt}_{k-1}(h) = n - \nu(h)$.

In [5], the formula

$$\sum_{c \in D_r} \mathrm{wt}(c) = q^{r-1}\, \mathrm{wt}_r(D_r) \tag{7}$$

is obtained. Then, putting $r = k$ and $r = k - 1$ in (7), we could immediately obtain (4), but in Sec. 2 we preferred to give another proof since it can easily be extended to a more general case as we will see below.

3.3. As has already been noted, the incidence relation $c \sim h$ of a point and a hyperplane ($c \in h$ in $\mathbb{P}C$, or $h \in c$ in $\mathbb{P}C^*$) is a particular case of an incidence relation between elements of two homogeneous spaces in duality [7, 8]. Let X and H be such a pair of homogeneous spaces in duality and ν be a function on H. Similarly to (1), we may consider the "anti-Radon transform"

$$R\colon \nu(h) \longmapsto (R\nu)(x) = \sum_h \nu(h) - \sum_{h \sim x} \nu(h). \tag{8}$$

In fact, for our purposes, homogeneity is not necessary. It only suffices to require that the incidence relation satisfies the conditions that the numbers $\#\{x \in X \mid x \sim h_0\}$ and $\#\{x \in X \mid x \sim h,\ x \not\sim h_1\}$ do not depend on h_0 and h_1, $h_1 \neq h_0$. Then, repeating the arguments used for deducing formulas (2) and (3), one easily obtains an analog of formula (4) for R^{-1}:

$$(R^{-1}f)(h) = \frac{\sum\limits_{x} f(x)}{\#\{x \in X \mid x \sim h_0\}} - \frac{\sum\limits_{x \sim h} f(x)}{\#\{x \in X \mid x \sim h_0,\ x \not\sim h_1\}}. \tag{9}$$

Let us give an example.

Example 2. Let $H = \mathbb{P}C^*$, $X = G(r, C) \cong G(k-r, C^*)$, and $x \sim h \Leftrightarrow h \in x$ (in the last expression, x is regarded as a plane of codimension r in $\mathbb{P}C^*$; cf. Sec. 3.2). Then, according to (6), the transform (8) precisely defines the rth generalized weight of a subcode x, i.e., $R\nu(x) = \mathrm{wt}_r(x)$. To distinguish this case from the general one, we denote this transform by R_r. Thus, we consider the transform $R_r\colon \nu \mapsto \mathrm{wt}_r$. Then (9) immediately yields the following result.

Proposition 2 (inversion formula for generalized weights). *Let C be a nondegenerate linear code of dimension k. Assume that for each of its r-subcodes D, where $1 \leq r \leq k-1$, the generalized weight is known. Then C is* *reconstructed up to equivalence as follows:*

$$\nu(h) = \frac{\sum_{D} \mathrm{wt}_r(D)}{q^{k-r}\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}} - \frac{\sum_{D\subset h} \mathrm{wt}_r(D)}{q^{k-r-1}\begin{bmatrix} k-2 \\ r-1 \end{bmatrix}}, \tag{10}$$

where $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_q = \dfrac{(q^a-1)\dots(q^a-q^{b-1})}{(q^b-1)\dots(q^b-q^{b-1})}$ *is the Gaussian binomial coefficient. Moreover, the first term on the right-hand side of* (10) *equals the length* n *of the code.*

Corollary. *If all* r*-subcodes* $(1 \le r \le k-1)$ *of a nondegenerate linear code have the same weight* d_r*, then the code is a* $d_r \dfrac{q-1}{(q^r-1)q^{k-r}}$ *times replicated symplex code.*

Note that the very fact of the possibility to reconstruct a code given the generalized weights follows from (7) and Proposition 1; however, (9) immediately gives the explicit formula.

Problem 2. In (10), it is assumed that we know wt_r as a function on $G(r, C)$ and the incidence relation $(D \subset h)$ is given. But if we only know the function $\mathrm{wt}_r\colon G(r, C) \to \mathbb{Z}$ as a function on a variety (say, on the image of the Plücker embedding) but do not know the incidence relation, can we reconstruct C up to equivalence? More precisely, what is the set of equivalence classes of codes that can be reconstructed in such a way from a function on a Grassmann variety?

Another example of using the formula (9) will be given below (Remark 2 and Proposition 3).

3.4. In some sense (though implicitly), the question on the inverse transform (4) was studied in [3]; in [4], the ideas of [3] are exposed in detail and in maximum generality and many examples are given. More precisely, in [3] it is proposed to consider the function wt of a code (or a function $a\,\mathrm{wt}+b$) as multiplicities for a new code. In our notations, this means the study of the transforms $R(R\nu)$ and $R(aR\nu + b)$. Remarkably, this gave some record-breaking results for ternary codes. As for the inverse transform, it appeared in these works in the following sense: it was shown that $\forall a, b\, \exists a', b'$ such that

$R(a'R(aR\nu + b) + b') = R\nu$ (see [3, Sec. 4] "Going back and forth" and [4, Sec. 5.2] "The inverse of the dual transform").

Much more interesting and promising idea considered in these works is to replace the degree-one polynomial $at + b$ with an arbitrary polynomial $P(t)$, i.e., to consider $R(P(R\nu))$ as a weight function of a code. It was found that if the coset weight distribution of the dual code $C^\perp$ is known, it is possible to compute the weight spectrum of the code thus constructed. In this way, a number of record-breaking codes is obtained in [3, 4].

4 Generalized weights as a weight function

4.1. Exterior algebra and the Plücker embedding. Let us first briefly recall the necessary notions and results. Let V be a k-dimensional vector space over a field F. Fix a basis in V. For an arbitrary set $v_1, \ldots, v_r$ of vectors of V, $r \le k$, consider the $r \times k$ matrix $(v_{\alpha\beta})$ whose rows are composed of the coordinates of these vectors in this basis, i.e., $v_{\alpha\beta}$ is the βth coordinate of v_α. Let $I(r,k)$ be the set of multiindices

$$\left\{\boldsymbol{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r,\ 1 \le i_1 < \cdots < i_r \le k\right\}$$

with, say, lexicographic order. Let

$$m_{\boldsymbol{i}}(v_1, \ldots, v_r) = \text{the } \boldsymbol{i}\text{th minor of } (v_{\alpha\beta}) = \det\left(v_{\alpha, i_\beta}\right)_{1 \le \alpha, \beta \le r}.$$

The vector $(m_{\boldsymbol{i}}(v_1, \ldots, v_r))_{\boldsymbol{i} \in I(r,k)}$ where $\boldsymbol{i}$ runs through all multiindices, i.e., a vector of length $\binom{k}{r}$, is called the *exterior product* of the vectors $v_1, \ldots, v_k$ and is denoted by $v_1 \wedge v_2 \wedge \ldots \wedge v_r$ ($\wedge$ is the operation of exterior multiplication). Obviously, exterior multiplication is linear in each of the arguments. It is also evident that $v_1 \wedge \ldots \wedge v_r = 0$ if and only if vectors $v_1, \ldots, v_r$ are linearly dependent. The vector space of dimension $\binom{k}{r}$ where these exterior products lie is called the *r-th exterior power* of V and is denoted by $\Lambda^r V$, its elements are called *r-vectors*, and the elements of the dual space $\Lambda^r V^*$ are called *exterior r-forms*. In particular, $\Lambda^1 V = V$, $\Lambda^0 V = \Lambda^k V = F$. Of course, by far not each r-vector is *decomposable*, i.e., can be represented in the form $v_1 \wedge \ldots \wedge v_r$, but any r-vector can be represented as a sum of decomposable ones. Since each minor can be expanded with respect to the first row, the operation $v \wedge \omega$ is well defined for $v \in V$, $\omega \in \Lambda^r V$. In the general case, for $\omega_1 \in \Lambda^{r_1} V$, $\omega_2 \in \Lambda^{r_2} V$, the product $\omega_1 \wedge \omega_2 \in \Lambda^{r_1+r_2} V$ is also well defined. In particular, this defines the duality $\Lambda^r V \cong \Lambda^{k-r} V^*$.

In the coordinate-free language, exterior algebra can be defined as the algebra of skew-symmetric tensors or as the Clifford algebra with zero scalar product.

We say that a *kernel* of $\omega \in \Lambda^r V$ is the set $\ker \omega$ of all $v \in V$ such that $v \wedge \omega = 0$. The following simple properties will be of use for us.

1. $v \wedge \omega = 0$ if and only if $\omega' \in \Lambda^{r-1} V$ exists such that $\omega = v \wedge \omega'$ (see, e.g., [10, Proposition 2]).
2. $\ker \omega$ is a linear subspace of V.

These two properties immediately imply the following one.

3. Each ω can be represented in the form $\omega = e_1 \wedge \ldots \wedge e_s \wedge \omega_0$, where $\ker \omega_0 = 0$ and $e_1, \ldots, e_s$ is a basis of $\ker \omega$.

Furthermore, it is clear that for any two bases $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ of a subspace $W \subset V$, the r-vectors $v_1 \wedge \ldots \wedge v_r$ and $w_1 \wedge \ldots \wedge w_r$ are proportional since each minor is multiplied by the determinant of the transition matrix. Thus, to each r-subspace W corresponds a unique point $P(W)$ in the projective space $\mathbb{P}\Lambda^r V$. This mapping $P\colon G(r, V) \to \mathbb{P}\Lambda^r V$ defined on the set $G(r, V)$ of all r-subspaces of V is injective and is called the *Plücker embedding*; for the structure of an algebraic variety on $G(r, V)$ (Grassmann variety, or Grassmannian), one may refer, e.g., to [9, 6].

4.2. Generalized weights of exterior forms. Let us consider the function wt_r defined on planes of codimension r in C^* (see (6)), i.e., on the Grassmannian $G(k-r, C^*)$. More precisely, consider it as a function on the image of the Plücker embedding, i.e., on a subset of the projective space $\mathbb{P}\Lambda^{k-r} C^* = \mathbb{P}^{\binom{k}{r}-1}$. We want to extend it over the entire projective space and then interpret the result as a weight function of some code. In other words, we want to introduce a function $\mathrm{wt}_{r,\Lambda}\colon \mathbb{P}\Lambda^{k-r} C^* \to \mathbb{Z}$ such that $\mathrm{wt}_{r,\Lambda}\big|_{G(k-r,C^*)} \equiv \mathrm{wt}_r$ and apply the transform R_1^{-1} to it.

A naive idea—to extend wt_r assigning to it a constant value outside $G(k-r, C^*)$—does not work since computation of R_1^{-1} according to (1) involves hyperplane sections of the Grassmannian, and these sections (in all cases except for the simplest ones) have rather complicated structure and the number of different types of sections is large (cf. [11]). However, we propose a quite natural way to extend wt_r on the entire space such that $R_1^{-1}\,\mathrm{wt}_r$ can be computed explicitly.

Note once more that under the Plücker embedding, to a plane h^r of codimension r in $\mathbb{P}C^*$ corresponds a point $\overline{\omega} \in \mathbb{P}\Lambda^{k-r}C^*$ such that $\omega \in \Lambda^{k-r}C^*$ is a decomposable form, i.e., $\omega = e_1 \wedge \ldots \wedge e_{k-r}$, where $\{e_i\}$ is a basis in h^r. The condition $h \notin h^r$ in the definition (6) of the generalized Hamming weight can be equivalently formulated in the language of exterior algebra as $h \wedge e_1 \wedge \ldots \wedge e_{k-r} \neq 0$, i.e.,

$$\mathrm{wt}_r(h^r) = \sum_{h\colon\, h\wedge\omega\neq 0} \nu(h).$$

It is quite natural to extend this definition to arbitrary exterior forms by putting

$$\mathrm{wt}_{r,\Lambda}(\overline{\omega}) \stackrel{\text{def}}{=} \sum_{h\colon\, h\wedge\omega\neq 0} \nu(h). \tag{11}$$

This formula defines the transform $R_{r,\Lambda}\colon \nu \mapsto \mathrm{wt}_{r,\Lambda}$.

Remark 2. This is another example of a transform of type (8), where the incidence of h and $\overline{\omega}$ means $h \wedge \omega = 0$. Note that the formula (9) for the inverse transform is valid in this case. Indeed, by property 1 of Sec. 4.1, all ω such that $\omega \wedge h_0 = 0$ are the forms of the type $h_0 \wedge \omega'$ where $\omega' \in \Lambda^{k-r-1}(C^*/\langle h_0\rangle)$, i.e., such ω form a linear space of dimension $\binom{k-1}{k-r-1} = \binom{k-1}{r}$. Therefore,

$$\#\{\overline{\omega} \mid \omega \wedge h_0 \neq 0\} = \theta_{\binom{k}{r}-1} - \theta_{\binom{k-1}{r}-1} = q^{\binom{k-1}{r}}\theta_{\binom{k-1}{r-1}-1}.$$

Analogously, all ω such that $\omega \wedge h_0 = 0$ and $\omega \wedge h_1 = 0$, where $h_1 \neq h_0$, are the forms of the type $h_0 \wedge h_1 \wedge \omega'$, $\omega' \in \Lambda^{k-r-2}(C^*/\langle h_0, h_1\rangle)$ and they form a linear space of dimension $\binom{k-2}{r}$. Therefore,

$$\begin{aligned}
&\#\{\overline{\omega} \mid \omega \wedge h_0 \neq 0,\ \omega \wedge h_1 = 0\} \\
&= \#\{\overline{\omega} \mid \omega \wedge h_1 = 0\} - \#\{\overline{\omega} \mid \omega \wedge h_1 = 0,\ \omega \wedge h_0 = 0\} \\
&= \theta_{\binom{k-1}{r}-1} - \theta_{\binom{k-2}{r}-1} = q^{\binom{k-2}{r}}\theta_{\binom{k-2}{r-1}-1}.
\end{aligned}$$

Thus, from (9) we obtain the following inversion formula.

Proposition 3 (inversion formula for generalized weights of exterior forms).

$$\nu(h)=\frac{\sum\limits_{\overline{\omega}}\mathrm{wt}_{r,\Lambda}(\overline{\omega})}{q^{\binom{k-1}{r}}\theta_{\binom{k-1}{r-1}-1}}-\frac{\sum\limits_{\overline{\omega}:\,\omega\wedge h=0}\mathrm{wt}_{r,\Lambda}(\overline{\omega})}{q^{\binom{k-2}{r}}\theta_{\binom{k-2}{r-1}-1}},$$

where $\overline{\omega}\in\mathbb{P}\Lambda^{k-r}C^*\cong\mathbb{P}\Lambda^rC$ *and* ω *is any form from* $\Lambda^{k-r}C^*$ *corresponding to* $\overline{\omega}$*.*

Problem 3. Here, we may also put a question similar to Problem 2: If we know $\mathrm{wt}_{r,\Lambda}$ as a function on a projective space $\mathbb{P}^{\binom{k}{r}-1}$ but do not know the structure of the exterior power, what is the set of equivalence classes of codes that can be thus reconstructed?

Let us return to our basic idea—to regard the obtained weights $\mathrm{wt}_{r,\Lambda}$ on the space $\mathbb{P}\Lambda^{k-r}C^*$ as ordinary weights for some long linear code $\boldsymbol{C}$, that is, to consider $\widetilde{\nu}=R_1^{-1}R_{r,\Lambda}\nu$. Let ν_h be the indicator function of a point $h\in\mathbb{P}C^*$. Then ν can be decomposed as $\nu=\sum\limits_h\nu(h)\nu_h$. Since the transforms $R_{r,\Lambda}$ and R_1^{-1} are linear in ν, it suffices to apply $R_1^{-1}R_{r,\Lambda}$ to ν_h. Then, by the definition (11), we have

$$(R_{r,\Lambda}\nu_h)(\overline{\omega})=\begin{cases}1, & \omega\wedge h\neq 0,\\ 0, & \omega\wedge h=0.\end{cases}$$

But, as we have seen above in Remark 2, the forms ω such that $\omega\wedge h=0$ form a linear space of dimension $\binom{k-1}{r}$, i.e., $R_{r,\Lambda}\nu_h$ is the indicator function of the complement to a plane of codimension $\binom{k-1}{r-1}$ in $\mathbb{P}\Lambda^{k-r}C^*$. Then we immediately obtain from Example 1 (Sec. 2) that

$$R_1^{-1}R_{r,\Lambda}\nu_h=\begin{cases}\dfrac{1}{q^{\binom{k-1}{r-1}-1}} & \text{on a plane of dimension } \binom{k-1}{r-1}-1 \text{ in } \mathbb{P}\Lambda^{k-r}C\cong\mathbb{P}\Lambda^rC^*,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$q^{\binom{k-1}{r-1}-1}R_1^{-1}R_{r,\Lambda}\nu_h=\begin{cases}1 & \text{at } \theta_{\binom{k-1}{r-1}-1} \text{ points,}\\ 0 & \text{otherwise.}\end{cases}$$

Since $\nu = \sum_h \nu(h)\nu_h$, this means that the function $q^{\binom{k-1}{r-1}-1} R_1^{-1} R_{r,\Lambda}\nu$ takes integer values, that is, corresponds to a linear code $\boldsymbol{C}$, and the length of $\boldsymbol{C}$ equals $n\theta_{\binom{k-1}{r-1}-1}$, where $n = \sum_h \nu(h)$ is the length of C.

Thus, we have obtained an $[N, K]$ code $\boldsymbol{C}$ with

$$N = n\theta_{\binom{k-1}{r-1}-1}, \qquad K = \binom{k}{r}.$$

Let us consider the weight distribution of this code and, in particular, find the minimum distance.

4.3. Weights. Spectrum. As follows from the construction, the set of weights of all codewords of the obtained code $\boldsymbol{C}$ coincides with the set of values of the function $q^{\binom{k-1}{r-1}-1}\,\mathrm{wt}_{r,\Lambda}$. Let us therefore consider the values of $\mathrm{wt}_{r,\Lambda}$.

Denote by $(\Lambda^{k-r}C^*)_s$ the set of all forms from $\Lambda^{k-r}C^*$ with an s-dimensional kernel (see Sec. 4.1, properties 1–3), i.e.,

$$\omega \in \left(\Lambda^{k-r}C^*\right)_s \iff \omega = e_1 \wedge \ldots \wedge e_s \wedge \omega_0,$$

where $\omega_0 \in (\Lambda^{k-r-s}C^*)_0$ is a kernel-free form. Obviously,

$$\Lambda^{k-r}C^* \setminus \{0\} = \bigsqcup_{s=0}^{k-r} \left(\Lambda^{k-r}C^*\right)_s. \tag{13}$$

Then for an arbitrary form $\omega \in (\Lambda^{k-r}C^*)_s$ we have

$$\#\{h \mid h \wedge \omega = 0\} = \#\{h \mid h \wedge e_1 \wedge \ldots \wedge e_s = 0\}$$

i.e.,

$$\mathrm{wt}_{r,\Lambda}(\omega) = \mathrm{wt}_{r,\Lambda}(e_1 \wedge \ldots \wedge e_s \wedge \omega_0) = \mathrm{wt}_{k-s,\Lambda}(e_1 \wedge \ldots \wedge e_s),$$

which means that $\mathrm{wt}_{r,\Lambda}(\omega)$ is the generalized Hamming weight of a $(k-s)$ subcode corresponding to the subspace $\ker\omega$ in C^*. Thus,

$$\mathrm{wt}_{r,\Lambda}\big|_{(\Lambda^{k-r}C^*)_s}(\omega) = \mathrm{wt}_{k-s}(\ker\omega), \quad s = 0, \ldots, k-r.$$

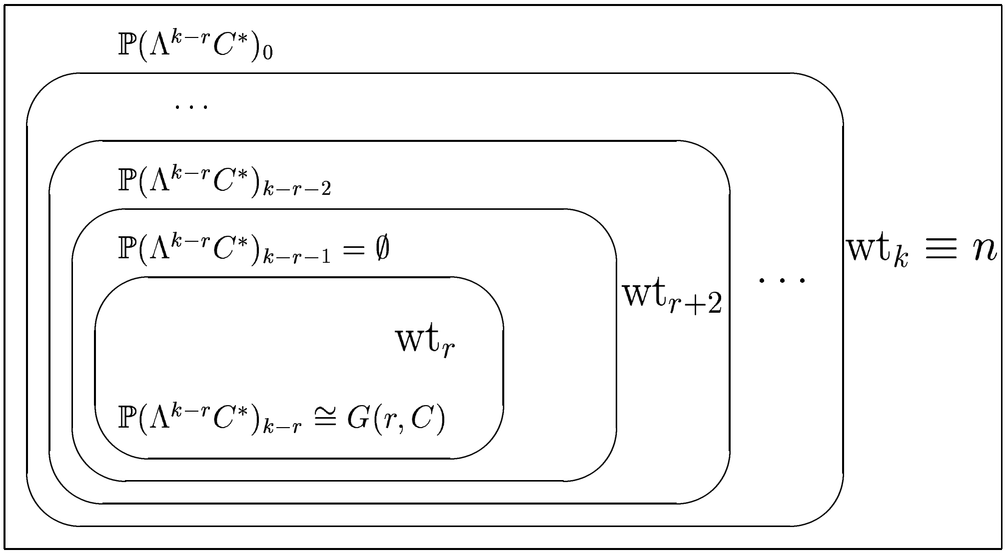

Fig. 1. The function $\mathrm{wt}_{r,\Lambda}$ on $\mathbb{P}\Lambda^{k-r}C^*$.

Schematically, this is presented in Fig. 1.

Therefore, weights of the codewords of $\boldsymbol{C}$ are the generalized weights of some subcodes of C of dimensions from r to k, multiplied by $q^{\binom{k-1}{r-1}-1}$. Of some subcodes, but not of all possible, since not for each s-subspace $V_s \subset C^*$ there exists a form $\omega \in \Lambda^{k-r}C^*$ such that $V_s = \ker\omega$ (this, of course, depends on s only, but not on a particular V_s). For a fixed $V_s = \langle e_1, \dots, e_r\rangle$, the number of ω such that $\ker\omega = V_s$, i.e., the number of different forms of the type $\omega = e_1 \wedge \dots \wedge e_s \wedge \omega_0$ where $\ker\omega_0 = \emptyset$, equals the number of forms $\omega_0 \in \left(\Lambda^{k-r-s}(C^*/V_s)\right)_0$. Thus,

$$\#\{\omega \in \Lambda^{k-r}C^* \mid \ker\omega = V_s\} = \left|\left(\Lambda^{k-r-s}(C^*/V_s)\right)_0\right|.$$

Let us introduce the following notation. For an arbitrary u-dimensional vector space V_u over $\mathbb{F}_q$, denote $x_u^p = |(\Lambda^{u-p}V_u)_0|$. In particular, as one can easily check, $x_p^p = q-1$ (kernel-free 0-forms, i.e., nonzero constants), $x_{p+1}^p = 0$ (all 1-forms, obviously, have kernels), $x_u^1 = 0$ (kernel-free linear functions), x_u^2

is the number of symplectic bilinear forms of rank u in a u-dimensional space; thus, $x_u^2 = 0$ for odd u.

In these notations,

$$\#\{\omega \in \Lambda^{k-r} C^* \mid \ker \omega = V_s\} = x_{k-s}^r \tag{14}$$

and hence

$$\left|\left(\Lambda^{k-r} V_k\right)_s\right| = \begin{bmatrix} k \\ s \end{bmatrix} x_{k-s}^r. \tag{15}$$

Thus, denoting $k - s = \rho$, we finally obtain that for each ρ-subcode $D \subset C$, $\rho = r, \ldots, k$, there exist exactly x_ρ^r codewords of $\boldsymbol{C}$ with weight $\mathrm{wt}_\rho(D) \cdot q^{\binom{k-1}{r-1}-1}$, and $\boldsymbol{C}$ has no other codewords. In particular, the minimum distance is $d_1(\boldsymbol{C}) = d_r(C) \cdot q^{\binom{k-1}{r-1}-1}$.

Lemma. *The numbers x_u^p are given by*

$$x_u^p = \sum_{j=0}^{u-p} (-1)^j \left(q^{\binom{u-j}{p}} - 1\right) \begin{bmatrix} u \\ j \end{bmatrix} q^{\binom{j}{2}}. \tag{16}$$

We derive this formula in the Appendix.

If, as usual, we denote by (A_w^ρ) the generalized weight spectrum of C, where A_w^ρ is the number of ρ-subcodes of weight w,

$$A_w^\rho = \#\{D \subset C \mid \dim D = \rho,\ \mathrm{wt}_\rho(D) = w\},$$

and denote by $W_{\boldsymbol{C}}(t) = \sum\limits_{i=0}^{N} A_i(\boldsymbol{C}) t^i$ the (ordinary) weight enumerator of $\boldsymbol{C}$ (i.e., A_i is the number of codewords of weight i), then the final result can be formulated as follows.

Theorem. *For any linear $[n, k]$ code C over $\mathbb{F}_q$ with generalized spectrum A_w^ρ and for any $r = 1, \ldots, k$, a linear $[N, K]$ code $\boldsymbol{C}$ exists with $N = n\theta_{\binom{k-1}{r-1}-1}$, $K = \binom{k}{r}$, and with weight enumerator*

$$W_{\boldsymbol{C}}(t) = 1 + \sum_{\rho=r}^{k} \sum_{w=d_\rho(C)}^{n} A_w^\rho x_\rho^r t^{wq^{\binom{k-1}{r-1}-1}},$$

where the numbers x^r_ρ are defined by (16). In particular, $d_1(\boldsymbol{C}) = d_r(C) \cdot q^{\binom{k-1}{r-1}-1}$.

4.4. Multiplicities. Thus, we have constructed the code $\boldsymbol{C}$ with multiplicities $\nu_{\boldsymbol{C}} = q^{\binom{k-1}{r-1}-1} R_1^{-1} R_{r,\Lambda}\nu$. We are interested in the number of repetitions, i.e., points where $\nu_{\boldsymbol{C}} > 1$. Therefore, let us examine what are the values of $\nu_{\boldsymbol{C}} \colon \mathbb{P}\Lambda^r C^* \to \mathbb{Z}$. To do this, let us return to the description of the transform $R_1^{-1} R_{r,\Lambda}$ given in Sec. 4.2 (formula (12) and below). As we have seen, for the indicator function ν_h of a point $h \in \mathbb{P}C^*$ the support of the function $R_1^{-1} R_{r,\Lambda}\nu_h$ is a plane of dimension $\binom{k-1}{r-1} - 1$ in $\mathbb{P}\Lambda^r C^*$. Let us describe this plane.

According to (12) and Example 1, this plane consists of the points $\overline{\omega'} \in \mathbb{P}\Lambda^r C^*$ that are incident to all $\overline{\omega} \in \mathbb{P}\Lambda^{k-r} C^*$ such that $\omega \wedge h = 0$. Since the incidence between points of $\mathbb{P}\Lambda^r C^*$ and $\mathbb{P}\Lambda^{k-r} C^*$ is defined by a nondegenerate pairing $\wedge$ on $\Lambda^r C^* \times \Lambda^{k-r} C^*$, we find that this plane consists of such $\overline{\omega'} \in \mathbb{P}\Lambda^r C^*$ that

$$\forall \omega \in \Lambda^{k-r} C^* \quad (\omega \wedge h = 0 \quad \Longrightarrow \quad \omega \wedge \omega' = 0).$$

Note now that if $h \in \ker \omega'$, i.e., $\omega' = h \wedge \omega''$, then this condition is fulfilled and such $\overline{\omega'}$ lie in the plane, and the linear space of all ω' such that $\omega' = h \wedge \omega''$ has precisely the dimension $\binom{k-1}{r-1}$. Hence,

$$\left(q^{\binom{k-1}{r-1}-1} R_1^{-1} R_{r,\Lambda}\nu_h\right)(\overline{\omega'}) = \begin{cases} 1, & h \wedge \omega' = 0 \\ 0, & h \wedge \omega' \neq 0. \end{cases}$$

Therefore, since $\nu = \sum\limits_h \nu(h)\nu_h$, we obtain that

$$\nu_{\boldsymbol{C}}(\overline{\omega'}) = \sum_{h \colon h \wedge \omega' = 0} \nu(h) = \sum_{h \in \ker \omega'} \nu(h) = n - \mathrm{wt}_{k-s}(\ker \omega'),$$

where $s = \dim \ker \omega'$, $0 \le s \le r$.

Thus, similarly to Fig. 1, we obtain the schematic description of $\nu_{\boldsymbol{C}}$ presented in Fig. 2. Here, similarly to (14), to each s-subspace $V_s \subset C^*$ correspond exactly $x^{k-r}_{k-s}/(q-1)$ points at which $\nu_{\boldsymbol{C}}$ takes the value $n - \mathrm{wt}_{k-s}(V_s)$.

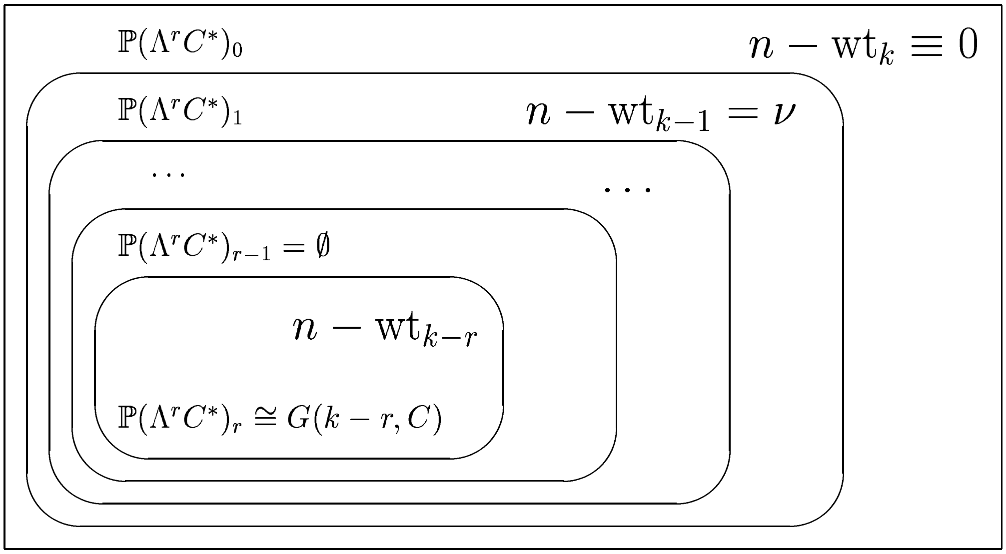

Fig. 2. The function $\nu_{\boldsymbol{C}}$ on $\mathbb{P}\Lambda^r C^*$.

Appendix

Proof of the Lemma. Counting the number of points in

$$\Lambda^{u-p} V_u \setminus \{0\} = \bigsqcup_{s=0}^{u-p} \left(\Lambda^{u-p} V_u\right)_s$$

(cf. (13)) and using (15), we find

$$x_u^p + \begin{bmatrix} u \\ 1 \end{bmatrix} x_{u-1}^p + \begin{bmatrix} u \\ 2 \end{bmatrix} x_{u-2}^p + \cdots + \begin{bmatrix} u \\ u-p \end{bmatrix} x_p^p = q^{\binom{u}{p}} - 1.$$

Replacing u by $u-1, u-2, \dots, u-p$, we thus obtain the system of equations

$$\begin{pmatrix} 1 & \genfrac{[}{]}{0pt}{}{u}{1} & \genfrac{[}{]}{0pt}{}{u}{2} & \dots & \genfrac{[}{]}{0pt}{}{u}{u-p} \\ & 1 & \genfrac{[}{]}{0pt}{}{u-1}{1} & \dots & \genfrac{[}{]}{0pt}{}{u-1}{u-p-1} \\ & & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ & & & 1 & \genfrac{[}{]}{0pt}{}{p+1}{1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} x_u^p \\ x_{u-1}^p \\ \dots\dots \\ x_{p+1}^p \\ x_p^p \end{pmatrix} = \begin{pmatrix} q^{\binom{u}{p}} - 1 \\ q^{\binom{u-1}{p}} - 1 \\ \dots\dots\dots \\ q^{\binom{p+1}{p}} - 1 \\ q - 1 \end{pmatrix}.$$

To find x_u^p, we need to compute the upper row of the inverse matrix. The jth entry of this row ($j = 0, \dots, u-p$) equals

$$(-1)^j \det \begin{pmatrix} \genfrac{[}{]}{0pt}{}{u}{1} & \genfrac{[}{]}{0pt}{}{u}{2} & \dots & \genfrac{[}{]}{0pt}{}{u}{j} \\ 1 & \genfrac{[}{]}{0pt}{}{u-1}{1} & \dots & \genfrac{[}{]}{0pt}{}{u-1}{j-1} \\ & 1 & \dots & \genfrac{[}{]}{0pt}{}{u-2}{j-2} \\ & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ & & 1 & \genfrac{[}{]}{0pt}{}{u-j+1}{1} \end{pmatrix}.$$

Using the notations $[a] = q^a - 1$ and $[a]! = [a][a-1]\dots[1]$, we rewrite the determinant as

$$\det\begin{pmatrix} \frac{[u]!}{[1]![u-1]!} & \frac{[u]!}{[2]![u-2]!} & \cdots & \frac{[u]!}{[j]![u-j]!} \\ \frac{[u-1]!}{[u-1]!} & \frac{[u-1]!}{[1]![u-2]!} & \cdots & \frac{[u-1]!}{[j-1]![u-j]!} \\ & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ & & \frac{[u-j+1]!}{[u-j+1]!} & \frac{[u-j+1]!}{[1]![u-j]!} \end{pmatrix}$$

$$= \frac{[u]![u-1]!\dots[u-j+1]!}{[u-1]![u-2]!\dots[u-j]!}\cdot\det\begin{pmatrix} \frac{1}{[1]!} & \frac{1}{[2]!} & \cdots & \frac{1}{[j]!} \\ 1 & \frac{1}{[1]!} & \cdots & \frac{1}{[j-1]!} \\ & \cdots\cdots & \cdots & \cdots\cdots \\ & & 1 & \frac{1}{[1]!} \end{pmatrix}$$

$$= \frac{[u]!}{[u-j]!}\cdot\frac{1}{[j]![j-1]!\dots[1]!}$$

$$\times\ \det\begin{pmatrix} [2][3]\dots[j] & [3]\dots[j] & \dots & [j-1][j] & [j] & 1 \\ [1][2]\dots[j-1] & [2]\dots[j-1] & \dots & [j-2][j-1] & [j-1] & 1 \\ & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots \\ & & [1][2][3] & [2][3] & [3] & 1 \\ & & & [1][2] & [2] & 1 \\ & & & & [1] & 1 \end{pmatrix}.$$

Now, subtracting the last column multiplied by [1] from the next to the last one and using the identity $[a]-[1]=[a-1]q$, we obtain

$$\frac{[u]!}{[u-j]![j]!}\cdot\frac{1}{[j-1]!\dots[1]!}\,q$$

$$\times\quad\det\begin{pmatrix} [2][3]\dots[j] & [3]\dots[j] & \dots & [j-1][j] & [j-1] \\ [1][2]\dots[j-1] & [2]\dots[j-1] & \dots & [j-2][j-1] & [j-2] \\ & \multicolumn{4}{c}{\dotfill} \\ & & [1][2][3] & [2][3] & [2] \\ & & & [1][2] & [1] \end{pmatrix}$$

$$= \quad \begin{bmatrix} u \\ j \end{bmatrix}\frac{1}{[j-2]!\dots[1]!}\,q$$

$$\times\quad\det\begin{pmatrix} [2]\dots[j-2][j] & [3]\dots[j-2][j] & \dots & [j] & 1 \\ [1]\dots[j-3][j-1] & [2]\dots[j-3][j-1] & \dots & [j-1] & 1 \\ & \multicolumn{4}{c}{\dotfill} \\ & & [1][3] & [3] & 1 \\ & & & [2] & 1 \end{pmatrix}.$$

Then, subtracting the last column multiplied by [2] from the next to the last one and using the identity $[a]-[2]=[a-2]q^2$, etc., we finally arrive at

$$\begin{bmatrix} u \\ j \end{bmatrix}\frac{1}{[1]!}q^{1+2+\dots+(j-1)}\det\begin{pmatrix} [1] \end{pmatrix} = \begin{bmatrix} u \\ j \end{bmatrix}q^{\binom{j}{2}}.$$

Therefore, the $(1,j)$th entry of the inverse matrix equals

$$(-1)^j\begin{bmatrix} u \\ j \end{bmatrix}q^{\binom{j}{2}},$$

whence (16) follows.

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399